J. Fluid Mech. (2003), *vol.* 494, *pp.* 399–405.
DOI: 10.1017/S0022112003006128 Printed in the United Kingdom

On the simple actuator disk

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(Received 5 April 2003 and in revised form 23 July 2003)

The standard textbook model of a helicopter rotor in vertical translation, a disk loaded with a uniform pressure jump in inviscid fluid, is revisited in search of correct descriptions of the far-field velocity and of the vortex sheet, allowing a rigorous control-volume analysis. The translation rate is not required to be large compared with the induced velocity. The classical results for induced power are unchanged, and now have a strong foundation: they are exact within the steady inviscid problem statement, instead of depending on a quasi-one-dimensional approximation as in the literature. Conversely, even with a uniform pressure jump the induced velocity is far from uniform over the disk, again in conflict with common beliefs and with any quasi-one-dimensional argument: the flow is upwards near the rim, both inside and outside it. The cross-section of the vortex sheet probably begins with a 45° spiral, as opposed to the smooth funnel shape that has been sketched, in the literature and below. A viscous numerical solution supports this conjecture. Plausible boundaries between the translation rates that produce the two 'clean' streamtube flow types, namely climb/hover and rapid descent, and those in-between that produce the vortex-ring state are also discussed.

1. Introduction

In the literature on helicopters (Gessow & Myers 1952; Bramwell 1976; Stepniewski 1979; Leishman 2000) the streamtube model (STM) of rotor flow fields combined with 'momentum theory' is presented as a slightly crude approximation, most useful for scaling arguments, but in the end providing better quantitative accuracy than could be expected. The derivations appear sketchy in places, and partly disconnected from the equations of motion. An obstacle is that for a rotor in and near hover, in contrast with a propeller, there is no small parameter on which to build a systematic approximation. The solution is also presented as producing 'unphysical' branches; this is unavoidable since the real flow follows one branch in climb and hover (C-H), and another in rapid descent (RD), which necessitates a 'jump'. It is tempting to call the conditions for descent rates between the ranges where the STM is supported by experiments the vortex-ring state. There is consensus on this, but not on whether the C-H branch becomes 'unphysical' in descent no matter how slow the descent or only for finite descent rates (i.e. of the same order as the average induced velocity v_h). The line type in figures often switches from solid in climb to dashed in descent, as a reminder. Considering the safety implications and that experimental and numerical results for induced power follow the C-H branch quite closely even for appreciable descent rates, nearing $v_h/2$, revisiting this régime is justified.

A weakness of the textbook derivations is that they are confined to the interior of the streamtube that crosses the actuator disk. Oddly, the control-volume analyses fail to mention the pressure forces on this tube, except for one passing comment

(Leishman 2000, p. 39). Since it has a non-uniform cross-section, these pressures must enter an axial momentum balance. Knowledge of the velocity and pressure outside the tube and particularly in the far field is needed for rigorous momentum-balance statements, especially since the streamtube shape is not simple near the hover condition. This knowledge will be helpful for numerical studies as well, both in terms of validation and of devising effective far-field numerical boundary conditions. Finally, when attempting to take into account the turbulence which develops in the wake and thus erodes the validity of the inviscid model, it becomes essential to know the flow direction outside the streamtube.

Another issue is that the presentations usually imply that the induced velocity is uniform over the disk, at least to a good approximation for a uniformly loaded rotor. However, this is not confirmed by numerical solutions, and experiments are very unlikely to settle the issue because of the small number of blades. As this uniformity has been linked to the concept of an 'ideal rotor' with minimum power requirement, this failure could be important. The desirability of this uniform-load feature in practice suggests that actual aircraft are not far from achieving it; in other words, it is not only the simplest case to analyse. Misleading expectations, such as this one of a uniform velocity, will also confuse the validation of CFD efforts. Finally, the flow direction near the rotor edge has implications for blade–vortex interactions.

This note presents a more rigorous flow model and exact integral results, but falls short of a full analytical solution of the equations. The resulting performance equations are the same as before due to cancellations, so that the study does not challenge established scaling arguments nor design practices. The aim of this note is to improve understanding of a classical subject.

2. Flow model and principal results

2.1. *Problem statement*

Consider an actuator disk with radius R and pressure jump Δp in the x-direction. The device may be a rotor, propeller, or windmill; for helicopters, the x-axis will be identified with the upward or downward vertical direction, depending on the flight régime. The disk is translating at a velocity $-U$ in the x-direction with respect to the ambient fluid. The quantities Δp or U may be of either sign. Climb, hover and descent cases are treated with the same equations, as are propellers and windmills, save for the final selection of a root of a quadratic equation. However it is assumed that the action of the disk creates a vortex tube that exits in the positive x-direction. This depends on the combination of Δp and U and the conditions under which it appears physical will be discussed later. In the reference frame of the disk the flow is steady, and the velocity in the far field is U. This is seen in figure 1, where the streamtube shape is assumed to be smooth for now, and the easily visualized climb (or propeller) case is taken.

For large positive x, the streamtube has radius R_2, and the velocity inside it, with reference to the frame of the disk, is $U + u_2$. The velocity of the vorticity in the shear layer surrounding the streamtube is $U + u_2/2$. Therefore, a first-cut criterion for the model to be physical is $U + u_2/2 > 0$. This is taken up again in § 2.4.

There is no assumption that the velocity through the disk is nearly aligned with the x-axis, nor uniform. In fact the only assertion is that the edge of the actuator surface is the circle of radius R; the surface could be a dome or cone, without changing the velocity field. This is because only the curl of the force field has any influence in an incompressible fluid, and this curl is concentrated on the rim.

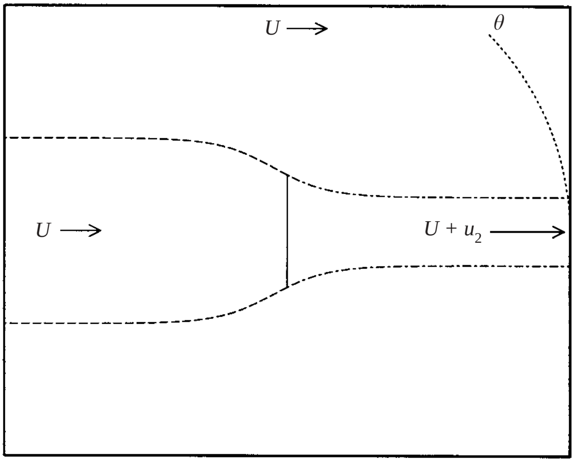

FIGURE 1. Flow schematic in a helicopter climb or propeller condition: —, disk; - - -, streamtube, not carrying vorticity; - — -, streamtube, carrying vorticity. U and $U+u_2$ indicate the approximate local velocity, with respect to the disk's reference frame.

2.2. *Far-field velocity*

The disk is aspirating fluid and expelling it through the tube at a rate $\pi R_2^2 u_2$ over the ambient U. Other than the tube surface, the fluid is irrotational. Therefore, the flow field induced by the disk at large distances is a sink, of magnitude $\pi R_2^2 u_2$, so that the velocity vector at a distance $r \gg R$ is

$$\boldsymbol{u} = U\boldsymbol{e}_x - u_2 \frac{R_2^2}{4r^2}\boldsymbol{e}_r + O\left(\frac{1}{r^3}\right); \tag{1}$$

here $\boldsymbol{e}_x$ and $\boldsymbol{e}_r$ are the unit vectors in the x- and r-directions. The spherical coordinates place $\theta = 0$ as the direction in which the outflow tube points, so that $\boldsymbol{e}_r \cdot \boldsymbol{e}_x = \cos\theta$. To leading order, the sink term is what bends the streamline in figure 1. The leading correction of order $1/r^3$ corresponds to an offset of the sink along the x-axis; it is centred at a distance of order R from the origin. The reason is that the shape of the streamtube is not known, and it represents the placement of a distribution of vortex rings in the vicinity of the origin, with a finite total circulation. The velocity induced by these vortex rings is of order $1/r^3$ for large r.

In rapid climb or rapid descent, the streamtube of the actuator disk extends far upstream and far downstream, in keeping with the sketches in the literature and figure 1. In contrast, in hover or slow climb and descent, the streamtube feeds from a sleeve around itself and not from above. The air rises, and turns around before draining through the tube. This is due to the sink term in the velocity formula (1).

2.3. *Performance equations*

The control volume is a large sphere. The x-momentum entering a sphere is the integral of $-(\rho u_x \boldsymbol{u} + p\boldsymbol{e}_x)\cdot\boldsymbol{e}_r$, with ρ the density and p the pressure. Bernoulli's equation $p + \rho|\boldsymbol{u}|^2/2 = p_\infty + \rho U^2/2$ applies, with an additional Δp applied to the fluid that went through the disk. The pressure p_2 in the tube at $x \gg R$ matches the pressure outside the tube at $r = x$ with $\theta \to 0$; therefore, the pressure integral can be taken over the complete sphere. In contrast, the integrals for the momentum term $\rho u_x \boldsymbol{u}$ are separated. The limits of integration for the sphere boundary outside the outflow tube are $\arcsin(R_2/r) < \theta < \pi$. The momentum leaving through the tube is $\pi R_2^2\rho[U+u_2]^2$. The momentum input due to the disk is $\pi R^2\Delta p$. Integrals in the azimuthal direction give 2π, the flow being axisymmetric. This leads to

$$-2\pi r^2\rho\int_{\arcsin(R_2/r)}^{\pi} u_x u_r \sin\theta\,\mathrm{d}\theta - 2\pi r^2\int_0^{\pi} p\boldsymbol{e}_x\cdot\boldsymbol{e}_r\sin\theta\,\mathrm{d}\theta + \pi R^2\Delta p = \pi R_2^2\rho(U+u_2)^2.$$

Substitute expressions for u and p from (1) and Bernoulli's equation, accurate to $O(1/r^3)$:

$$-2\pi r^2\rho\int_{\arcsin(R_2/r)}^{\pi}\left(U-u_2\frac{R_2^2}{4r^2}\cos\theta\right)\left(U\cos\theta-u_2\frac{R_2^2}{4r^2}\right)\sin\theta\,\mathrm{d}\theta$$

$$-2\pi r^2\int_0^{\pi}\left(p_\infty+\rho U u_2\frac{R_2^2}{4r^2}\cos\theta\right)\cos\theta\sin\theta\,\mathrm{d}\theta + \pi R^2\Delta p = \pi R_2^2\rho(U+u_2)^2 + O\left(\frac{1}{r}\right).$$

The integrals give

$$R^2\Delta p = R_2^2\rho(U+u_2)u_2 \tag{2}$$

in which the $O(1/r)$ correction was dropped since r is arbitrarily large.

Bernoulli's equation for fluid far downstream in the streamtube is $p_\infty+\rho(U+u_2)^2/2 = p_\infty+\Delta p+\rho U^2/2$, or

$$\rho u_2\left(U+\frac{u_2}{2}\right) = \Delta p. \tag{3}$$

This quadratic yields u_2, from U and $\Delta p/\rho$. Combined with (2), it gives

$$\left(U+\frac{u_2}{2}\right)R^2 = R_2^2(U+u_2), \tag{4}$$

which yields R_2. The volume flow through the disk gives the total power $P = \Delta p\pi R^2(U+u_2/2)$, and the induced power

$$P_i = \Delta p\pi R^2\frac{u_2}{2}, \tag{5}$$

since the useful power is $\Delta p\pi R^2 U$.

These are the classical results, but a derivation based on the momentum balance in a large domain with a far-field behaviour known to a high enough order (here, $O(1/r^3)$) is far more satisfactory. There is no appeal to 'quasi-one-dimensional' arguments. It is not assumed, nor shown, that the induced velocity is uniform at the disk. Let $v_h \equiv \sqrt{|\Delta p|/(2\rho)}$ be the 'standard induced velocity in hover', as given by (3) and (4) with $U=0$ since $R^2 v_h = R^2 u_2$. In hover, the average induced velocity is indeed v_h, but numerical solutions show the velocity to be about $1.22\times v_h$ near the axis, and to be opposite (an upflow) near the edge of the disk, roughly for $r>0.92R$ (in an actual rotor, this tendency may be overpowered by the influence of the blade's bound vortex). As a result, the idea that the 'ideal rotor' has a uniform induced velocity at the disk, or even tends towards it, is erroneous. The superficial similarity with the 'ideal wing' in lifting-line theory is probably the cause of this long-lived misconception. Nevertheless, a rotor with uniform Δp does minimize the induced power for a given disk size and thrust. The reason is that the induced velocity far downstream in the tube is uniform at $U+u_2$ (the curl of the velocity being zero inside the tube, and x derivatives having gone to zero, we have $\partial u/\partial r = \omega_\theta = 0$), so that the flux of kinetic energy is minimized for a given thrust and rotor area. This minimizes power.

The agreement between the present results and the traditional ones raises the suspicion that the cancellation is a simple matter that could be and had been predicted, making the contribution redundant. However, to the author's knowledge, this did not

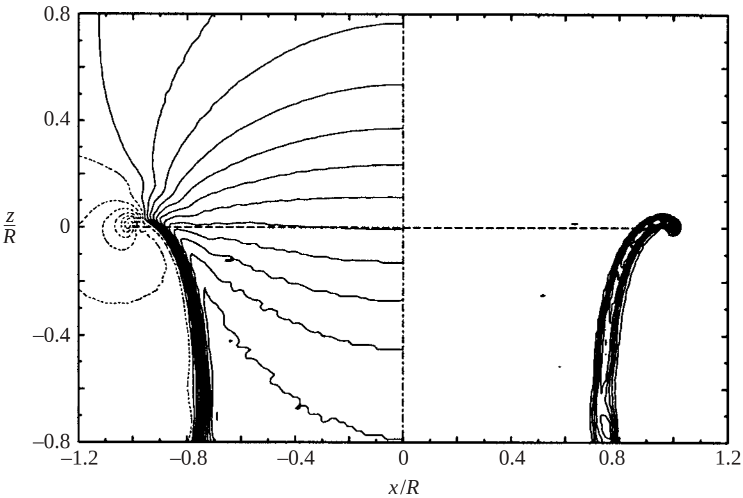

FIGURE 2. Flow field near actuator disk in hover, $U = 0$. Left, contours of axial velocity; —, down, - - - up. Right, contours of vorticity.

occur. The words 'one-dimensional' are ubiquitous in the textbooks, whereas no such assumption is needed here. The cancellation amounts to the following. In figure 1, consider the streamtube from $-\infty$ to $+\infty$, and the pressure on it. The tube has convex and then concave curvature, causing a drop and then a rise in pressure, relative to the ambient pressure p_∞; this is with $U \neq 0$. This allows for a cancellation in the axial momentum balance. However, since the tube shape is not known analytically, nor is the pressure, predicting that cancellation is far from immediate. The proof by control-volume analysis is quite elementary, but requires a statement about the asymptotic behaviour of the velocity and pressure fields for large r, as in (1), which is not found in the literature.

2.4. *Streamtube shape*

This shape is not as simple as in the usual sketches (or that in figure 1). Imagine that its cross-section were a smooth curve. The part that carries vorticity would be a vortex sheet, with a finite density, ending abruptly on the edge of the disk. Now the velocity at the tip of such a sheet is infinite, so that this geometry is ruled out. It is most likely that the exact shape, for the infinitely thin sharp-edged disk, is a volute wound around the disk's edge. The two-dimensional Euler equations admit an exact solution with a vortex sheet of uniform density placed on a 45° spiral. It has an infinite number of turns but finite length, and therefore circulation. The density of the vortex sheet is $\Delta u = 1.477\sqrt{\Delta p/\rho}$, only slightly higher than it is far downstream ($\sqrt{2}\sqrt{\Delta p/\rho}$); as a result, the two asymptotes of the solution can join smoothly. These numbers apply in hover. Analytical or numerical solutions to the full problem, with zero thickness (pure top-hat loading) were not found in the literature, and neither was any mention of a spiral. For smoother pressure-jump distributions, the shape tends towards a spiral without creating the singularity, but still explains the upflow near the edge. This holds even in climb and rapid descent, although the extent of the spiral would shrink; essentially, the flow field needs to respond to a singular injection of vorticity, and streamlines with bounded curvature cannot support that.

Figure 2 supports this model of a spiral; contrast with the notional shape in figure 1. It is a solution of the incompressible Navier–Stokes equation with a Reynolds number of 4×10^3 based on v_h and R; U is zero. It was obtained from a spectral method with a grid spacing equal to $R/64$. The source of vorticity was de-singularized with a Gaussian of the type $\exp(-r^2/\sigma^2)$ with $\sigma = R/50$. Both the contours of axial velocity and the locus of the vortex sheet reveal the upwash near the edge of the disk, as well as all along the outside of the vortex tube. This was somewhat unexpected in hover, before (1) was established. The locus is consistent with a smeared spiral.

3. Region of validity of the streamtube model

3.1. *Notation*

This is not an issue for propellers and windmills. This section reverts to the traditional notation for helicopters: V_c for the climb rate (therefore upwards, in the direction of the thrust) and v_2 for the velocity far down inside the streamtube. Equation (3) gives two solutions $u_2 = \pm\sqrt{U^2 + 2\Delta p/\rho} - U$. The C-H branch of the curve has $\Delta p > 0$ (x downwards), $V_c = U$, $v_2 = u_2$. The negative solution for u_2 is discarded because it makes the streamtube exit in the wrong direction. The RD branch has $\Delta p < 0$ (x upwards), $V_c = -U < 0$, $v_2 = -u_2$. Solutions to the quadratic equation for u_2 exist only if $U^2 \geqslant 2|\Delta p|/\rho$, and the correct branch for u_2 is that with a + sign, the 'windmill brake state'. The – sign solution has negative $U + u_2$, meaning it is aspirating air through the tube, which is unphysical.

The vortex-ring state occurs roughly in the region $[-2, -1]$ for V_c/v_h, and is loosely identified with a failure of the STM. The STM predicts a large increase of the induced power (5) when descending (Gessow & Myers 1952, p. 130), but not a reversal of the dependence of total power on descent rate, which would be a plausible criterion to define a severe control difficulty.

3.2. *Condition of vorticity propagation far downstream*

The tentative condition $U + u_2/2 > 0$ was mentioned above. On the C-H branch it does not give any limit on V_c, because of the relationships $U/v_h = 2v_h/u_2 - u_2/(2v_h)$, so that $U + u_2/2 = 2v_h^2/u_2 > 0$ always. The situation is similar for the RD branch; the solution also 'avoids' the line $U + u_2/2 = 0$. This condition is not helpful, other than in selecting the sign in the solution of the quadratic equation, in § 3.1.

3.3. *Condition of vorticity propagation near the rotor*

The average induced velocity at the rotor is $u_2/2$, so that the vorticity transport velocity is approximately $U + u_2/4$ instead of $U + u_2/2$. The equations then yield the following. The C-H branch has $U + u_2/4 = 2v_h^2/u_2 - u_2/4$. This becomes negative when $u_2 = \sqrt{8}v_h$, and therefore $V_c = U = -\sqrt{2}v_h$. This is closer to the experimentally found value for which the induced power strongly deviates from the C-H branch of the model.

3.4. *Impact of jet turbulence*

The flow at positive x is a jet, which will become turbulent and mix in practice, in contrast with a steady axisymmetric Euler solution. In rapid climb, this turbulence is transported away and has little impact, but in slow climb or any descent the aircraft must catch up with its own turbulence: some of the air ingested by the rotor has been mixed with air that had already passed through it. There is recirculation, implying at least diffuse turbulence and vorticity. In that sense, the STM loses validity near the hover condition once turbulence is acknowledged to exist. However, this reason is not given in the literature, and there is no special meaning to the hover condition in this respect. At lower descent rates the turbulence is weaker by the time it comes back and envelops the near-field, although it still exists.

3.5. *Conclusions relative to the region of validity*

These considerations unfortunately do not provide tight boundaries between 'streamtube states' and vortex-ring states, even for a greatly idealized rotor. Approaching the vortex-ring state from above, by increasing the descent rate, the rotor re-ingests turbulence that has had less and less time to decay. The STM fails gradually. Approaching the vortex-ring state from below, the RD branch stops at $V_c = -2v_h$, as the streamtube above the rotor expands to infinity. This failure of the STM is more abrupt than the failure from above, and experimental results may support this idea, in that the power measured experimentally indeed appears to jump at $V_c = -2v_h$ in some datasets.

4. Summary

This note presents a mathematical model of the flow induced by an actuator disk in axial translation which appears to be new, although simple, and is more accurate than those in a half-century of helicopter literature. Due to cancellations, the performance results are unchanged, but the common presumption that the streamtube has finite curvature and especially the assertion that the induced velocity is fairly uniform are both proven to be incorrect. This is confirmed by numerical results. A prediction is attempted of the boundaries of the vortex-ring state by considering the propagation of vorticity and turbulence, yielding only moderate success but never in contradiction with test results. The extension to flight in general translation will be considered, and numerical simulations are in progress.

The author thanks Dr L. Wigton for identifying the spiral solution, and Dr J. Forsythe for reviewing the manuscript.

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